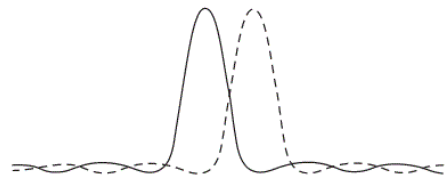
Example: Resolving power of a lens.

From the wave theory of light, the smallest angle a telescope can resolve is

$$\theta_{\rm diff} \simeq 1.22\frac{\lambda}{D}$$



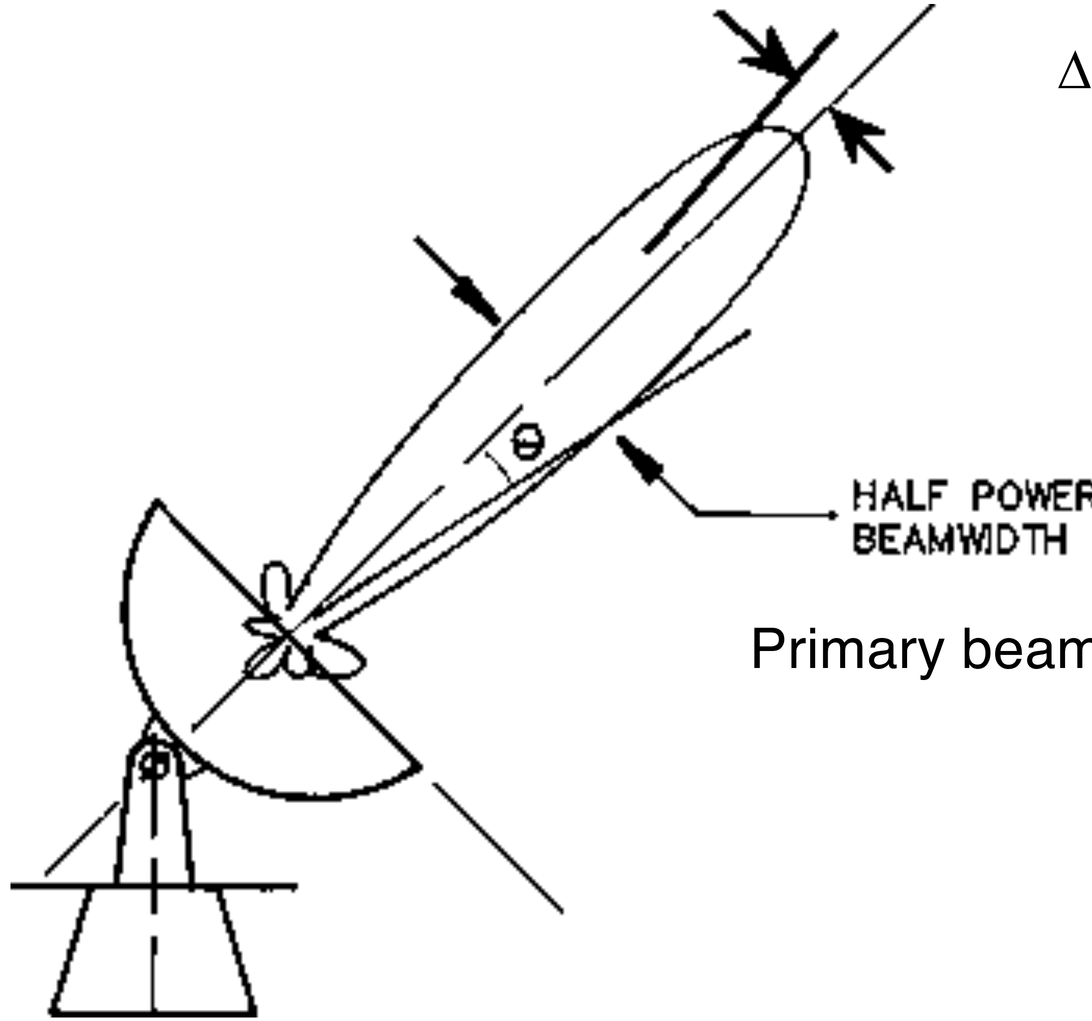
where D is the diameter of the telescope, and λ is the wavelength of the EM radiation.

We can derive this as a direct consequence of the uncertainty principle.

Antenna Beam Parameters

Pointing Accuracy

$\Delta\theta$ = rms pointing error



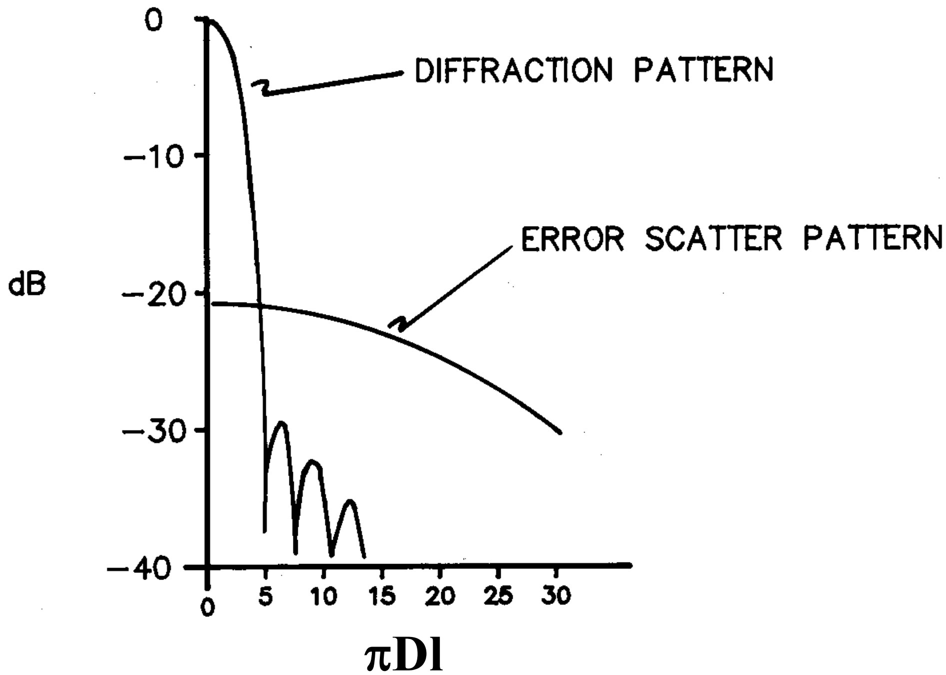
$\Delta\theta$

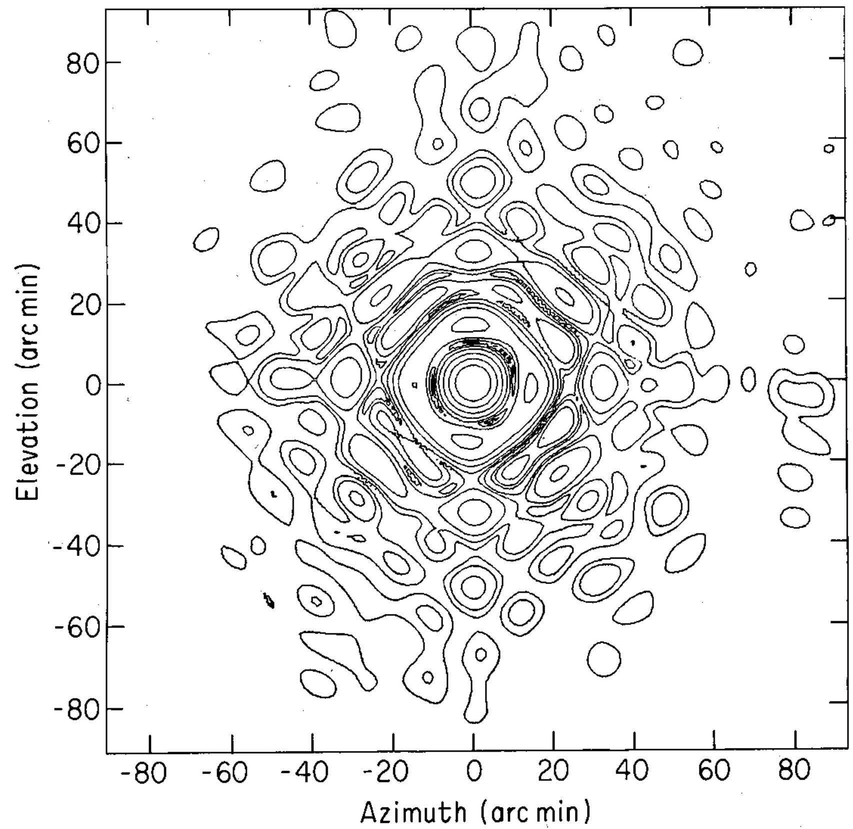
θ_{3dB}

Primary beam $\boldsymbol{A}(\theta)$

Beam Pattern

Primary Beam





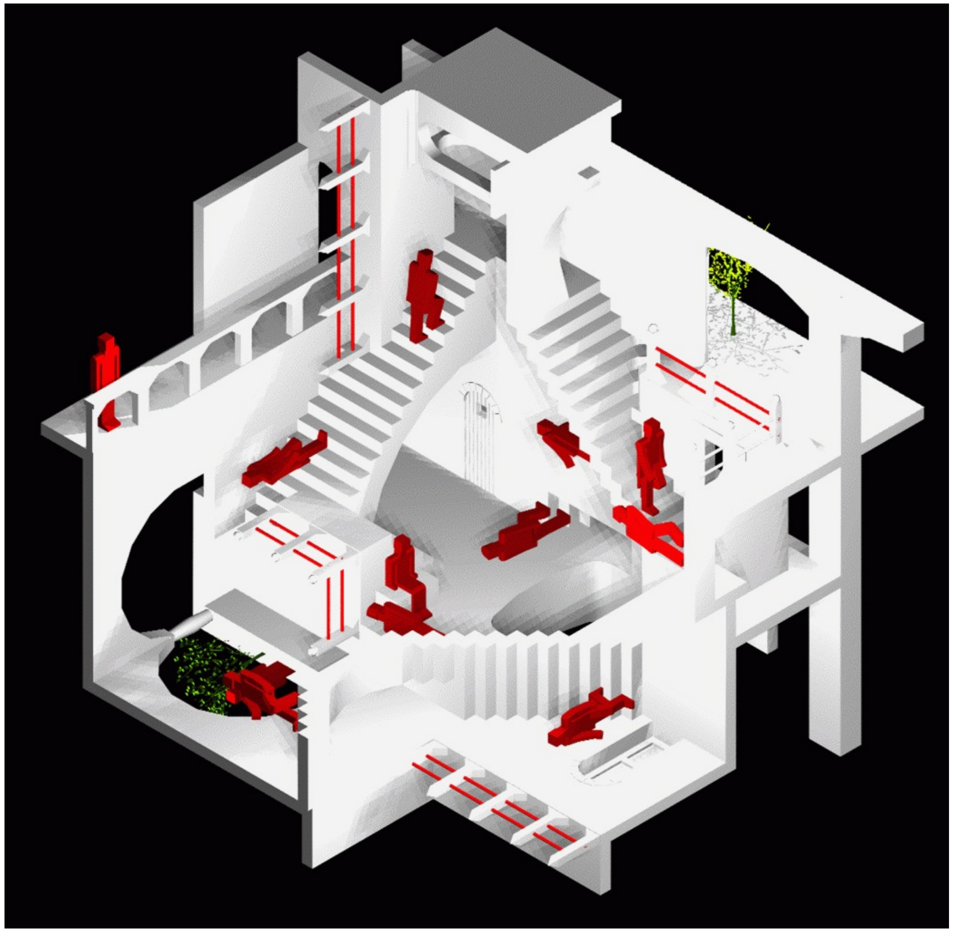
l=sin(θ), D = antenna diameter in wavelengths

contours:–3,–6,–10,–15,–20,–25, –30,–35,–40 dB

dB = 10log(power ratio) = 20log(voltage ratio)

For VLA: θ_{3dB} = 1.02/D, First null = 1.22/D

Astronomy 421



Lecture 5: Relativity

Outline:

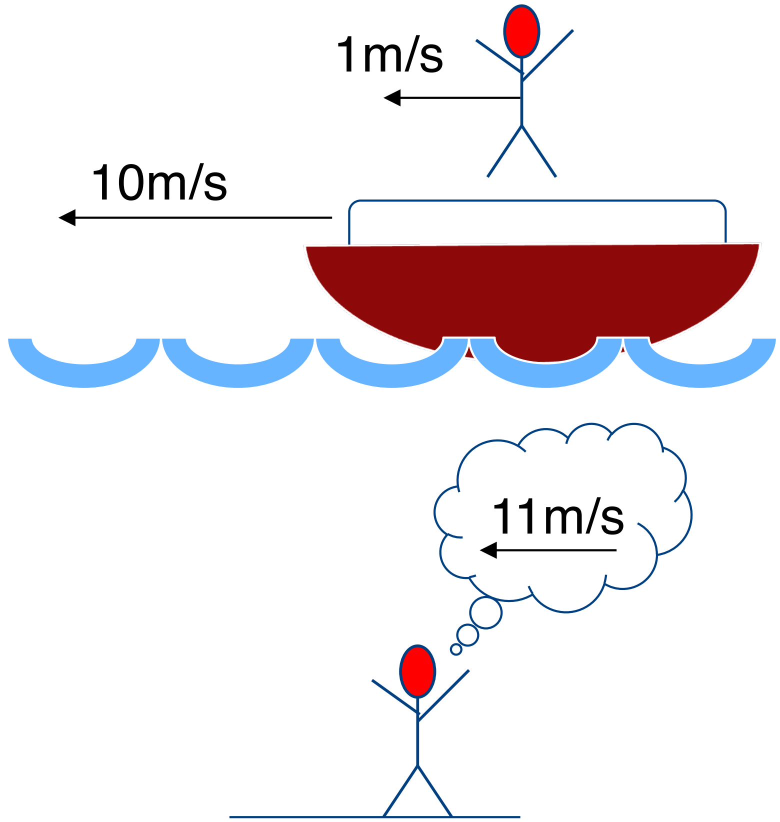
- Pre-Einstein History
- Einstein's Postulates
- Simultaneity, Time Dilation and Length Contraction
- Lorentz Transformations
- Spacetime

Galilean relativity

- Invariant: time, acceleration, force
- Relative: position, velocity

- Galilean transformation

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$



Special relativity

A bit of history:

1860 - James Clerk Maxwell's theory of electromagnetism predicted EM waves with $v=c$. He proposed that light was EM wave.

Big question: what does the waving for light waves? All other known waves required a *medium* to support vibrations, e.g., water, air, string.

Luminiferous ether was proposed as the required medium. EM waves move through ether with speed c.

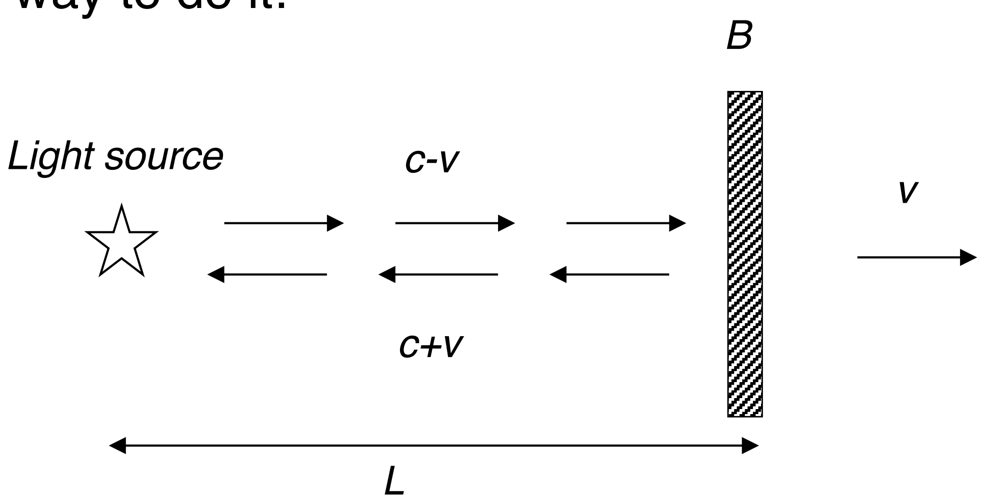
Present everywhere (even in space), massless but rigid to support high frequencies, yet no effect on planets since their orbits could be understood by Newton's laws with no ether drag.

The Michelson-Morley experiment (1887)

Idea was to detect Earth's motion relative to the ether. Presumably small because Earth's motions are << *c*.

If velocity of Earth relative to ether is *v*, and velocity of light wave relative to ether is *c*, then velocity of a light wave traveling in the same direction as Earth is *c-v* relative to Earth (upwind), and in opposite direction, *c+v*.

One way to do it:



Light pulse bouncing off mirror.

Entire system moving through ether at velocity *v*. What is prediction for time for light to return to source, which gives v?

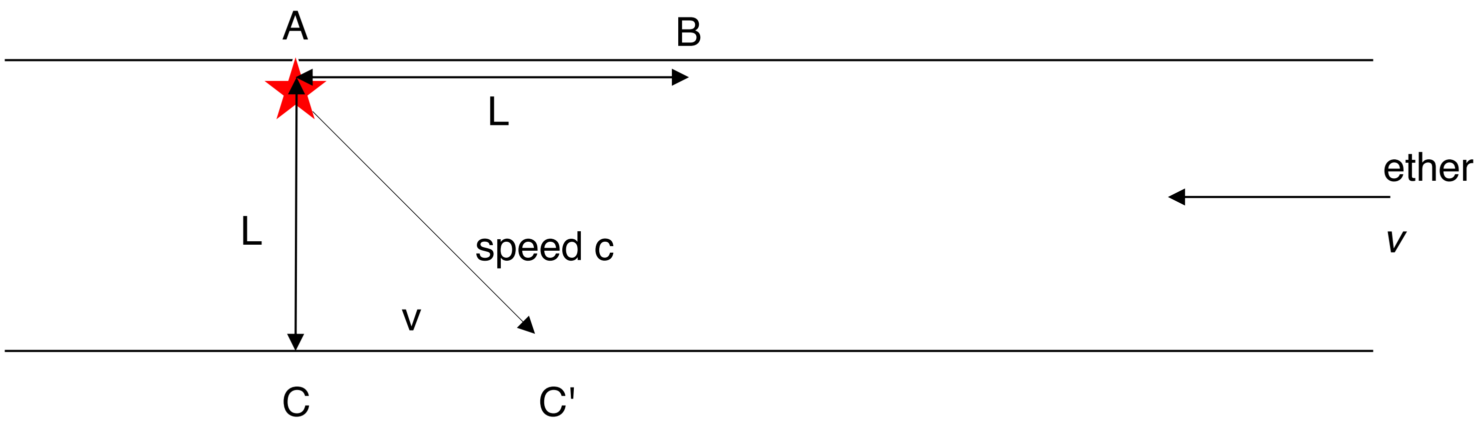
Total time under this assumption is thus:

$$T_1 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2cL}{c^2 - v^2} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

$$\text{since } \frac{v^2}{c^2} \ll 1, T_1 \simeq \frac{2L}{c}\left(1 + \frac{v^2}{c^2}\right)$$

If v = $v_\oplus \sim 30\,\mathrm{km\,s^{-1}}$, $\frac{v^2}{c^2} \sim 10^{-8}$

A very small effect. How to measure it?



Think of ether as river flowing to the left at speed *v*. For light (or a swimmer) to go from *A* to *B*, it goes "upstream" at velocity *c-v*. From *B* to *A*, it goes with current at *c+v*.

T_1 (time for A-B-A):

$$T_1 \simeq \frac{2L}{c}\left(1+\frac{v^2}{c^2}\right)$$

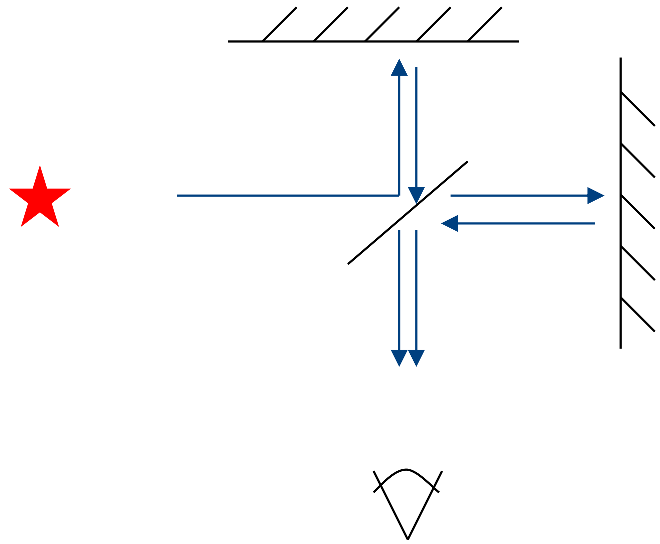
How about light reflected at point C? A swimmer has to head a bit upstream if she wants to reach C since current will pull her downstream. Light travels at c, current at speed v, so net speed across river is sqrt($c^2 - v^2$). Total time to cross river and back:

$$T_2 = \frac{2L}{\sqrt{c^2-v^2}} = \frac{2L}{c}\left(1-\frac{v^2}{c^2}\right)^{-1/2} \simeq \frac{2L}{c}\left(1+\frac{v^2}{2c^2}+...\right)$$

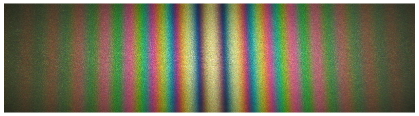
Different from T_1 . But still small, so how does this help?

Michelson-Morley experiment (1887):

Difference measurement employing interference of two light beams:



Observe fringe pattern of interfering light, rotate device 90 degrees and count number of fringes that shift due to changing time difference between paths.



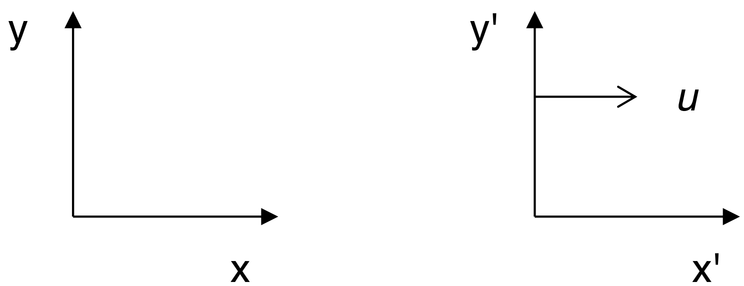
The difference in path lengths would lead to interference fringe maxima at certain positions. But hard to place elements accurately enough to measure true path lengths. So instead rotate the apparatus 90 degrees and measure shift in position of fringes as the path lengths change.

Null result: T's are always the same whether parallel or perpendicular to Earth's motion. Earth has no motion with respect to ether.

Coordinate Transformations

How do coordinates transform in general?

A. Before relativity, "common sense" Galilean transformations were assumed to be always valid.



S frame S' frame

S' moves at u along x-axis relative to S.

$x'=x-ut$	$x=x'+ut$
$y'=y$	$y=y'$
$z'=z$	$z=z'$
$t'=t$	$t=t'$

Einstein's postulates (1905):

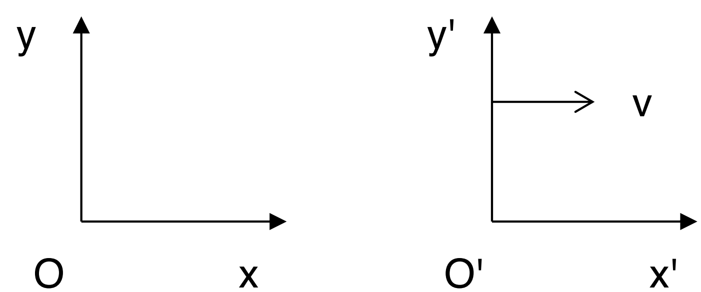
It was known that Maxwell's equations changed form under a Galilean transformation. Should laws of physics be allowed to vary for different reference frames, or is Galilean transformation not the whole story? Einstein says we need a transformation of space and time that preserves laws of physics in all frames.

1. The principle of relativity
   - the laws of physics are the same in all inertial reference frames.

2. The speed of light is constant
   - light travels through a vacuum at constant speed *c* that is independent of the motion of the light source. No ether needed, no speed relative to it.

Consequences: compare measurements made by observers who are moving wrt each other.

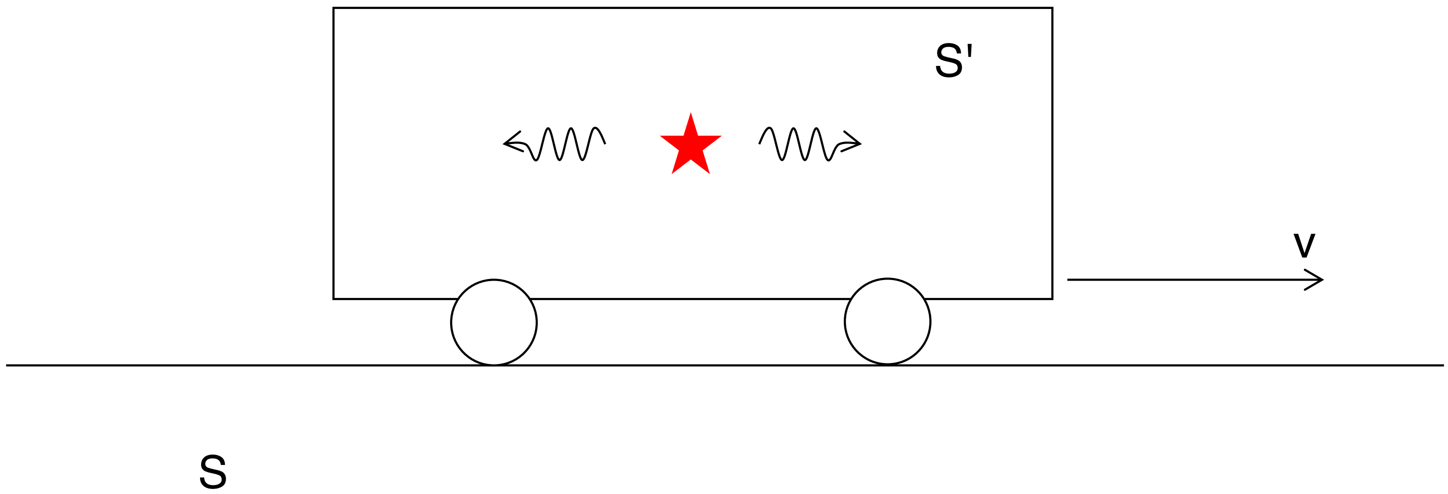
S' frame moves at velocity *v* wrt S, in *x* (or *x'*) direction.



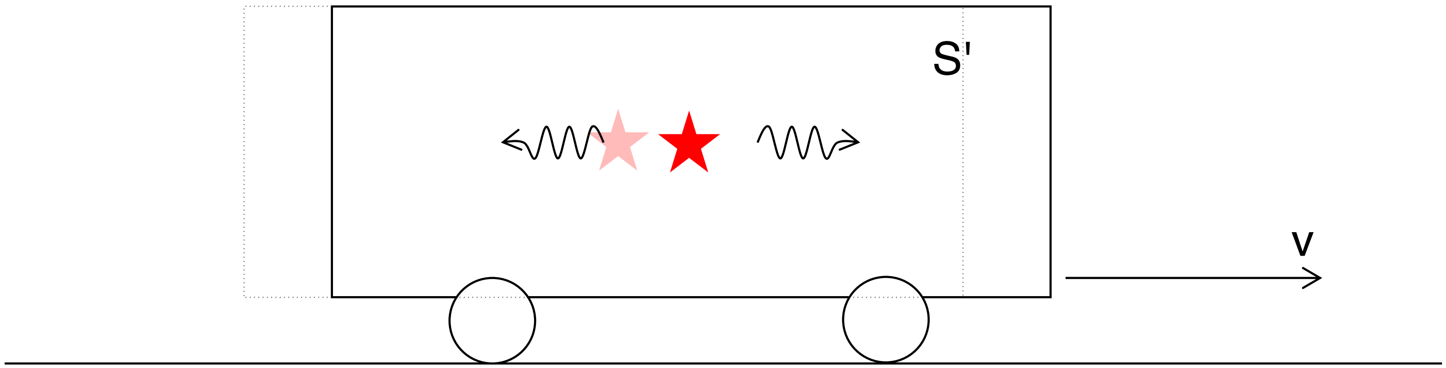
S frame S' frame

First Consequence: Loss of Simultaneity

Einstein's famous example - light flashing in railroad car. The bulb is in the middle of the car, observer S is stationary wrt tracks. Car moves to the right at velocity *v*.



The bulb flashes: according to S', light hits ends of car at same moment, because traveling at constant *c*, equal distances. Events are *simultaneous* in S' frame.



S

Observer S says: both pulses travel at c *relative to the tracks*. During the travel time of the light, the train moves to the right. Pulse going to left hits back end of train before right-going pulse hits front. Events are **not** simultaneous.

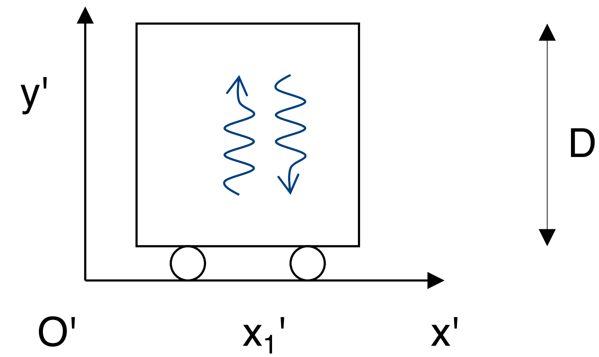
Who is right?

Both. Just as there is no absolute motion, there is no absolute simultaneity.

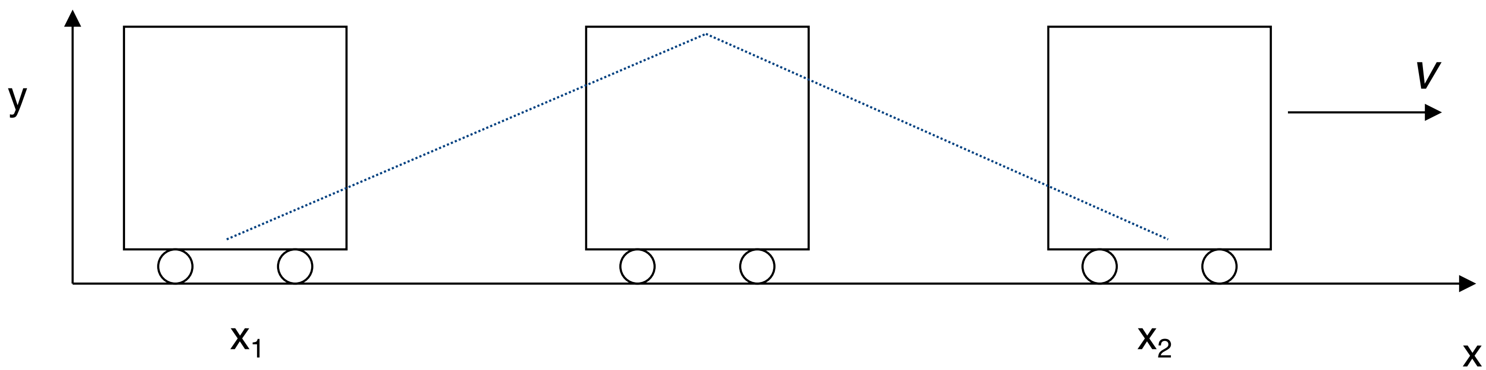
Simultaneity is relative, for events that occur at different spatial locations.

Second Consequence: Time Dilation

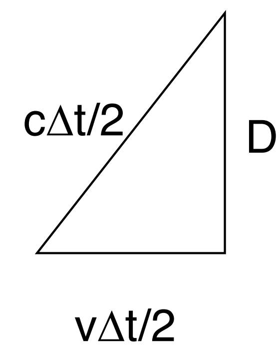
Consider the train again, with mirror on roof and vertical flashing.



S' frame



S frame



Observer at rest in S' flashes light and measure time taken $\Delta t'$ to hit roof and return. $\Delta t'=2D/c$.

What does this look like in S frame, where it takes a time Δt? Light path is longer in S. Solve for Δt from right triangle.

$$\frac{c^2\Delta t^2}{4} = \frac{v^2\Delta t^2}{4} + D^2$$

$$\Delta t = \frac{2D}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

also $\Delta t' = \frac{2D}{c}$

$$\Delta t = \frac{\Delta t'}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \gamma \Delta t'$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \geq 1$$

Lorentz factor

Observer in S frame claims clock in S' runs slowly since observe in S' claims a shorter time interval for the event. Observe in S' says S clocks are fast. If they agree on *c*, they give up time invariance.

The shorter time is measured in frame at rest relative to the events (here the beginning and end of the light's trip). This is "proper time".

How can we test this?

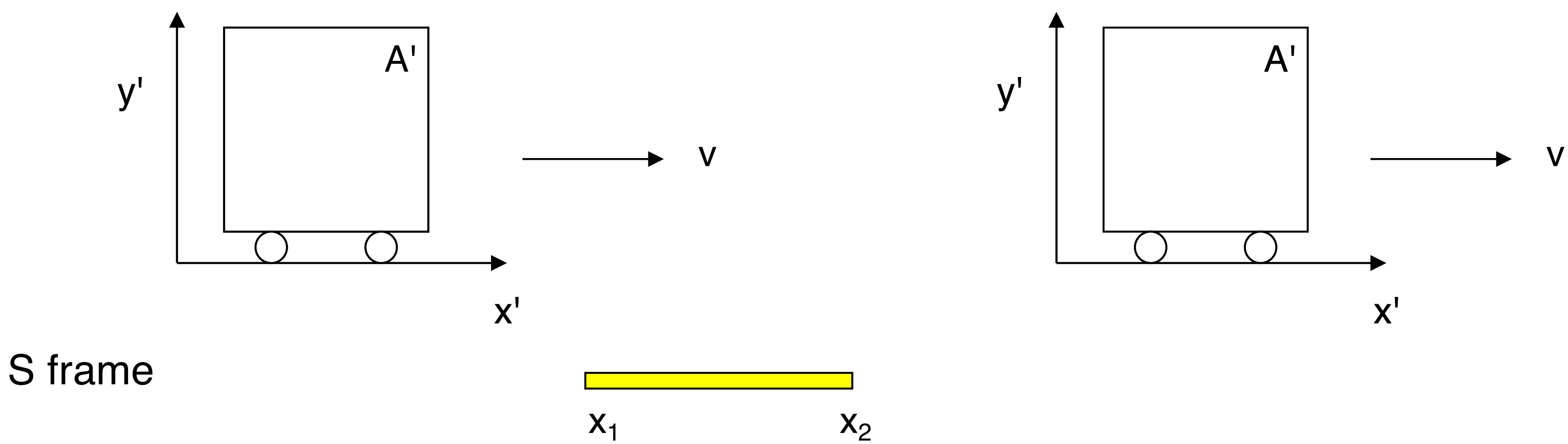
Effect is too small to see in daily life since $v \ll c$ normally.

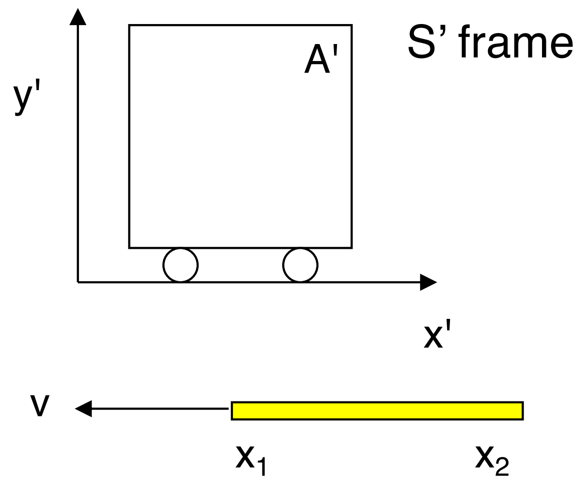
- We can test by taking two atomic clocks, and have one on an airplane flying around Earth.
  - *Moving* clock is found somewhat behind the one at *rest*, thus moving clocks run slower.

- Muons (elementary particle) are created when cosmic rays hit the upper atmosphere. Its lifetime is so short that most should decay before it hits the surface of the Earth - but many are detected!
  - Muons move with $v \sim c$ and their lifetime is longer in our frame, which is not the proper frame since beginning and end of muon life occurs in different places. Worksheet #4

Third Consequence: Length Contraction

Keep previous setup. Suppose x_1 and x_2 are the ends of a rod of length $L=x_2-x_1$, fixed in S. Observer in S sees that it takes a time Δt for train to pass by rod. So $L=v\Delta t$ according to observer in S.





In S', observer sees the rod moving past with speed *v* for time *Δt'*, so length in S' is *L'=v Δt'*.

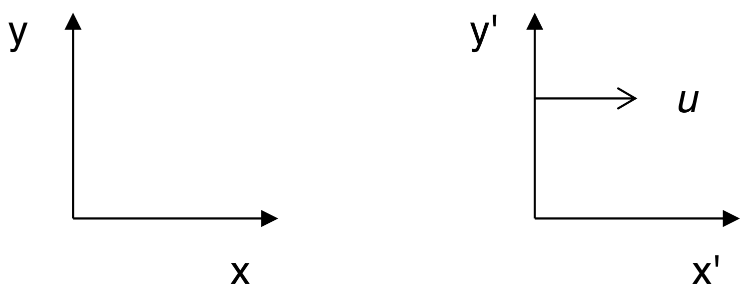
$$L' = v\Delta t\sqrt{1-\frac{v^2}{c^2}} = L\sqrt{1-\frac{v^2}{c^2}} = \frac{L}{\gamma}$$

The rod is smaller in a frame in which it is moving. The longest length called the *proper length* is measured in the rest frame of the rod.

In muon decay, muons see Earth's atmosphere contracted, so less distance to go during lifetime. So again, we see many at Earth's surface.

Back to Coordinate Transformations

A. Galilean transformations



S frame S' frame

S' moves at *u* along x-axis relative to S.

$x'=x-ut$	$x=x'+ut$
$y'=y$	$y=y'$
$z'=z$	$z=z'$
$t'=t$	$t=t'$

If something is at rest in S' frame at location *x'*, what is it doing in the S frame?

It moves away at velocity *u*. After time *t* it will be at *x=x'+ut* in S frame.

Take d/dt of above

$$\dot{x}' = \dot{x} - u \qquad \dot{x} = \dot{x}' + u$$

$$\dot{y}' = \dot{y} \qquad \dot{y}' = \dot{y}$$

$$\dot{z}' = \dot{z} \qquad \dot{z}' = \dot{z}$$

If light travels along x axis with speed *c* in S, this gives $\dot{x}' = c - u$ in S', violating second postulate of Special Relativity. Need transformation where speed is *c* in both frames.

B. Lorentz transformations (derived in 4.2 of C+O):

$$x = \gamma(x' + ut') \qquad x' = \gamma(x - ut)$$

$$y = y' \qquad y' = y$$

$$z = z' \qquad z' = z$$

$$t = \gamma(t' + \frac{ux'}{c^2}) \qquad t' = \gamma(t - \frac{ux}{c^2})$$

Consistent with Einstein's postulates, e.g. can show Maxwell's equations have same form in both frames. Can derive time dilation and length contraction from these (see C+O).

Velocity transformations:

Write above as Δ*x*, Δ*x*', or d*x*, d*x*' etc. and form *dx/dt* (=v_x) etc.:

$$v'_x = \frac{v_x - u}{1 - {}^{uv_x}/_{c^2}} \qquad v'_y = \frac{v_y\sqrt{1 - {}^{u^2}/_{c^2}}}{1 - {}^{uv_x}/_{c^2}} \qquad v'_z = \frac{v_z\sqrt{1 - {}^{u^2}/_{c^2}}}{1 - {}^{uv_x}/_{c^2}}$$

Can show: if $|\vec{v}| = c$, $\left|\overrightarrow{v'}\right| = c$. (C+O Prob. 4.12)

So distance between two events, and time between two events, both depend on observer's frame of reference. They are not *invariant*, unlike in Galilean transformations.

Is there some quantity involving both time and distance that is invariant?

Yes! (C+O 17.2)

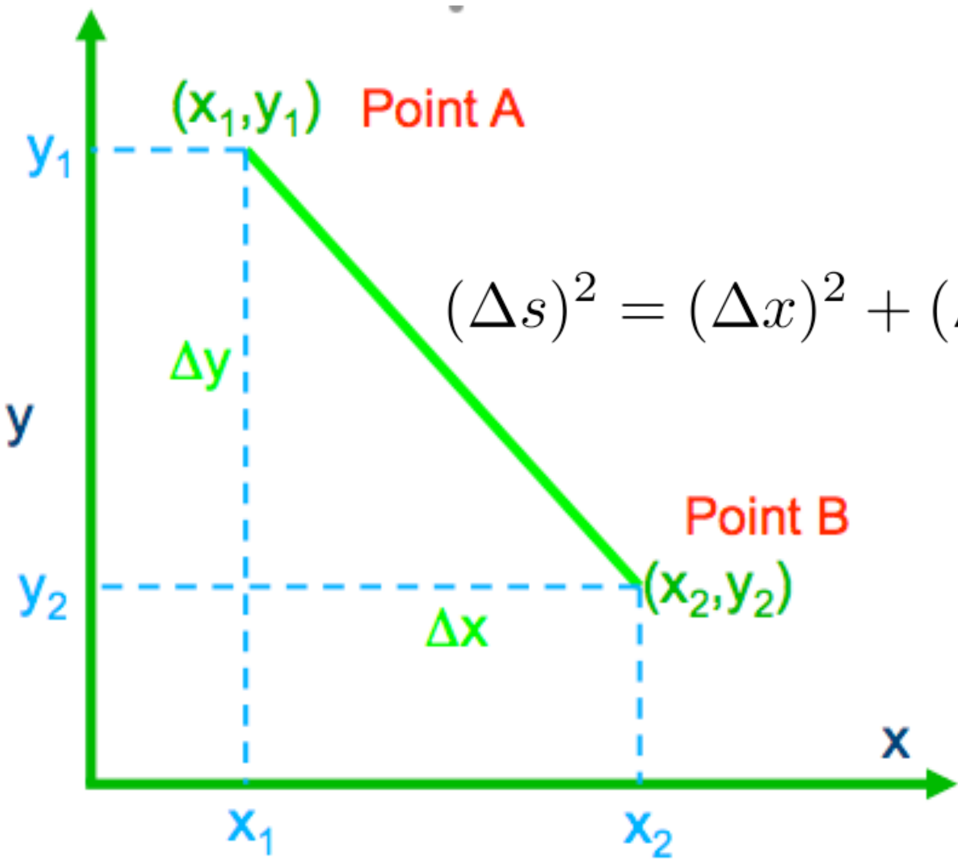
First, recall distance between two points in 3D:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

which is the distance between coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) with $\Delta x=x_2-x_1$, $\Delta y=y_2-y_1$ and $\Delta z=z_2-z_1$.

Cartesian intervals

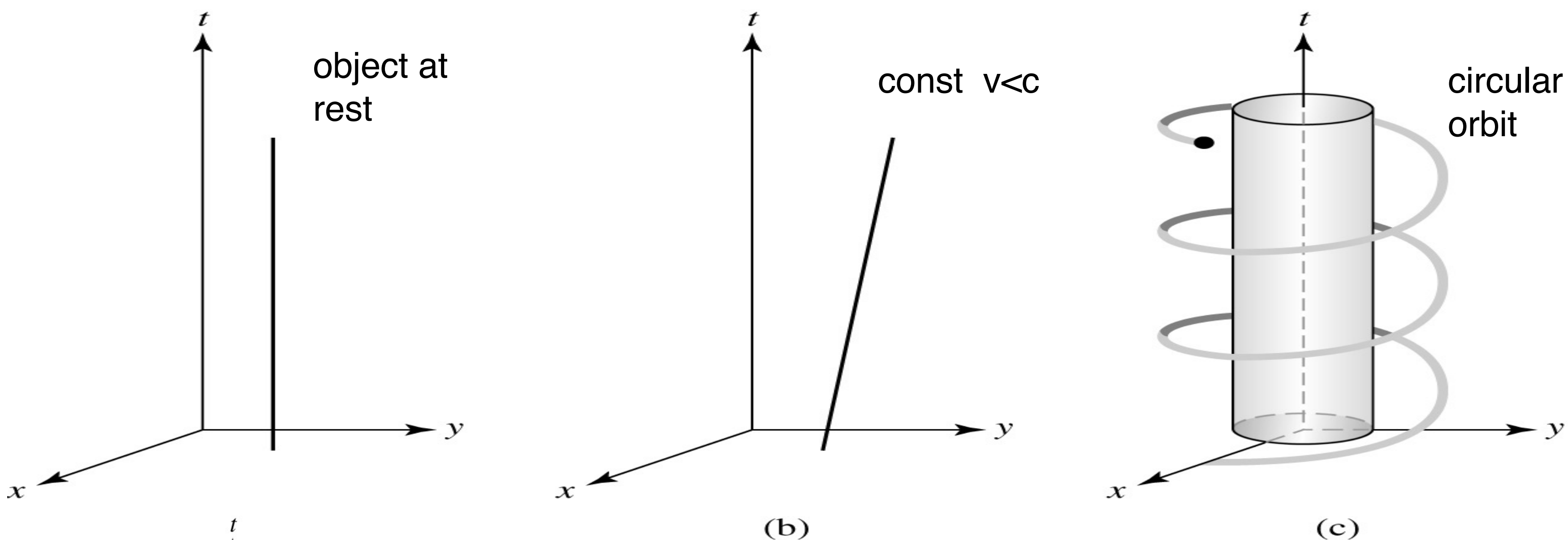
Δs^2 (length of a vector) is invariant under translation or rotation of the coordinate system (but is not invariant under a Lorentz transformation).

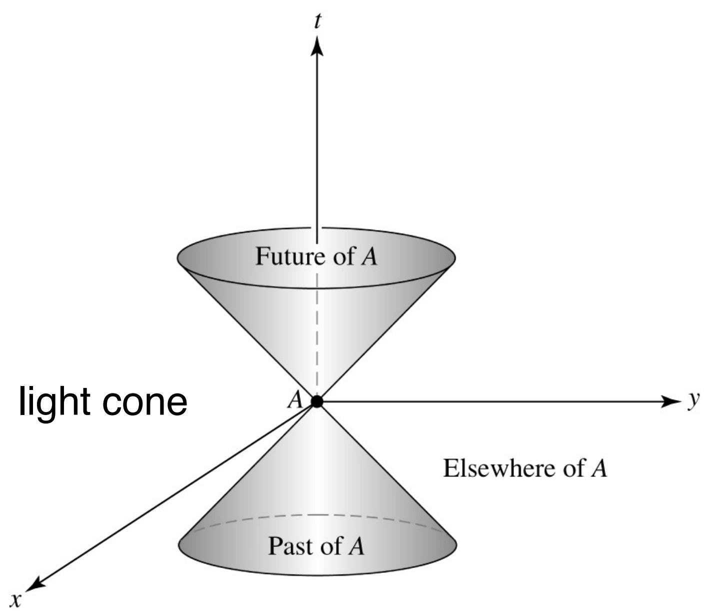


$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Spacetime Diagrams, Worldlines and Lightcones (see also C+O 17.2)

In relativity, must consider space and time together. Use spacetime diagrams and describe motion with worldlines





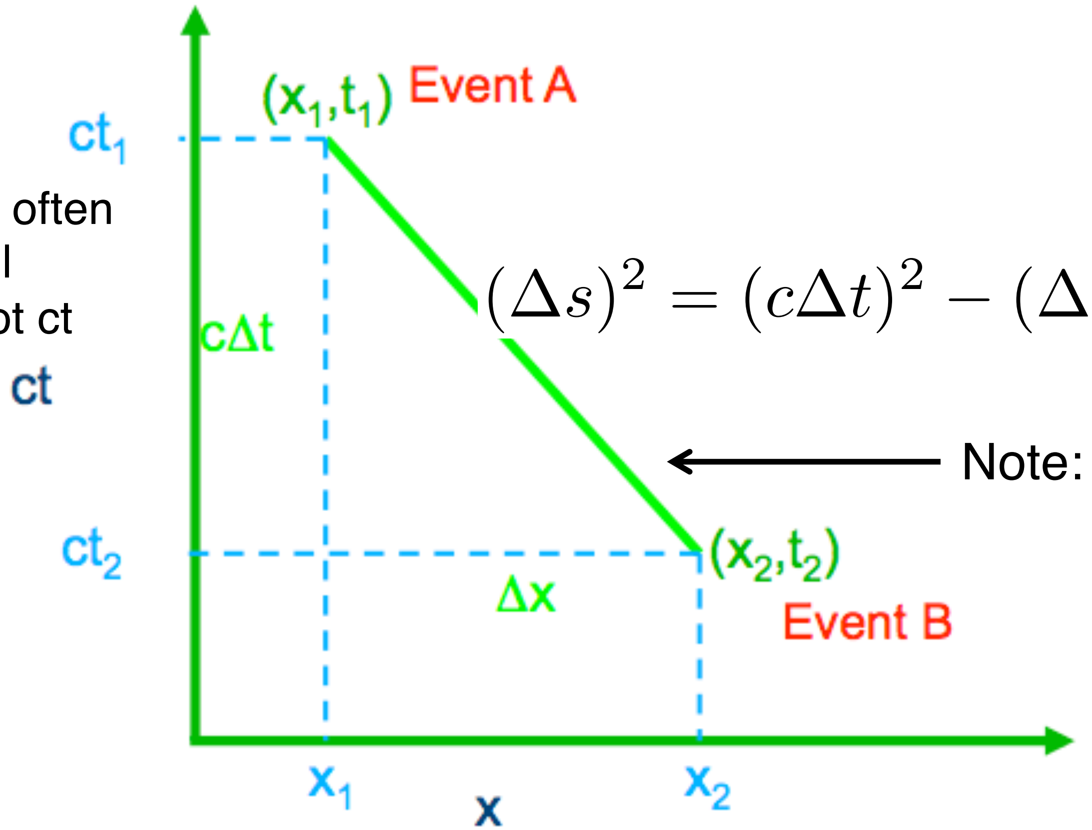
Light emitted at $x=y=t=0$.

Worldline obeys $ct = \sqrt{x^2 + y^2}$

=> somewhere on cone surface

Spacetime intervals

If we define the spacetime interval as below:



Note: often useful
To plot ct

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

Note: this line is not Δs!

Claim: Δs^2 is invariant under Lorentz transformations.

If Δs^2 is invariant between frames of reference, then:

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

$$= (\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

Confirm with Lorentz transformation with *Δx=x, Δy=y, Δz=z, Δt=t, Δx'=x', Δy'=y', Δz'=z', Δt'=t'.*

Need to show $$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2$$

$$\Delta s^2 = \Delta s'^2$$

$$\Delta s^2 = c^2\gamma^2(t' + \frac{vx'}{c^2})^2 - \gamma^2(x' + vt')^2 - y'^2 - z'^2$$

$$= c^2\gamma^2(t'^2 + \frac{2vx't'}{c^2} + \frac{v^2x'^2}{c^4}) - \gamma^2(x'^2 + 2vx't' + v^2t'^2) - y'^2 - z'^2$$

$$= \gamma^2 \left[c^2 t'^2 (1 - \frac{v^2}{c^2}) - x'^2 (1 - \frac{v^2}{c^2}) \right] - y'^2 - z'^2$$

$$= c^2 t'^2 - x'^2 - y'^2 - z'^2 = \Delta s'^2$$ QED

Interval Δs^2 , involving both space and time, is invariant. *Spacetime.*

Space is different for different observers.

Time is different for different observers.

Spacetime interval is the same for all observers.

Time-like, space-like and light-like intervals

- $\Delta s^2 = (c\Delta t)^2 - \Delta x^2 > 0 \quad \Rightarrow \quad c\Delta t > \Delta x$
  - spatial distance can be traveled with *v<c*

    => time-like interval

- $\Delta s^2 < 0 \quad \Rightarrow \quad c\Delta t < \Delta x$
  - spatial distance cannot be traveled with *v<c*

    => space-like interval

- $\Delta s^2 = 0 \quad \Rightarrow \Delta x = c\Delta t$
  - for a signal moving at the speed of light, in any frame

    => light-like interval

Relevant later for black holes

Definitions

- *Inertial reference frame:* a frame of reference in which the basic laws of physics apply (e.g. Newton's first law).
  - *Example:* a train moving at a constant velocity
  - What is an example of a non-inertial frame?

- *Inertial observer*: an observer in an inertial rest frame

- *Invariance:* a quantity is invariant if all inertial observers would obtain the same result from measuring the quantity.

- *Relativity:* a quantity is relative if different inertial observers obtain different results from measuring the quantity.