Lecture 2 – Key Concepts

- Newtonian mechanics and relation to Kepler's laws
- The Virial Theorem
- Tidal forces
- Collision physics

Announcements

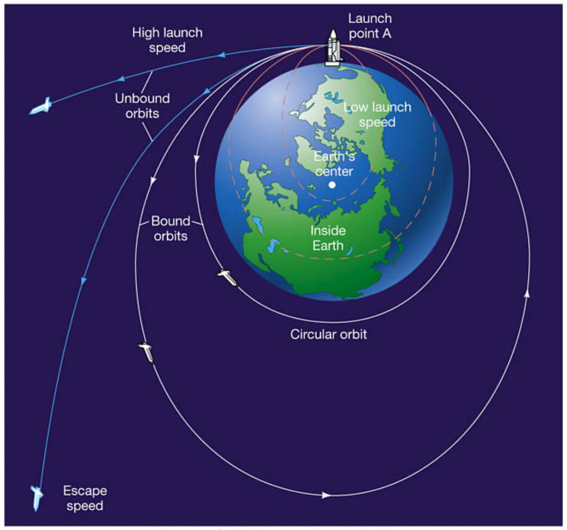
- Office hours – Wednesdays at 2pm or by appointment.
- Class web page:
- https://leo.phys.unm.edu/~gbtaylor/astr421/

Physical interpretation of Kepler's Laws

1. "A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse". Is it really at the focus?

   With Newton, the orbit shape depends on *E*, which is the sum of each body's KE plus the PE*:*

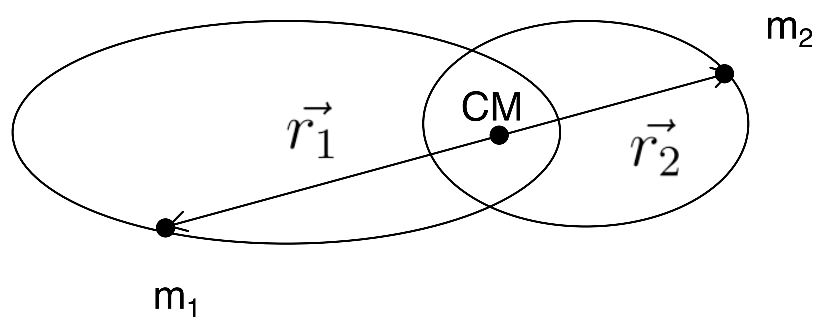
Total energy $E < 0$		circular or elliptical (bound)
	$= 0$	parabolic (marginal)
	> 0	hyperbolic (unbound)



- For two masses in bound orbit, orbits are ellipses with the center of mass at the common focus.

- Orbits easier to understand in center of mass reference frame.

The center of mass is always on a line between two objects:



C&O show that center of mass is unaccelerated if no external forces exist.

Define the *displacement* vector $\vec{r} = \vec{r_2} - \vec{r_1}$

Define reduced mass, μ , as $\mu = \frac{m_1 m_2}{m_1 + m_2}$

and the total mass $M = m_1 + m_2$

Can easily show: $\vec{r_1} = \frac{-\mu}{m_1}\vec{r}$ $\vec{r_2} = \frac{-\mu}{m_2}\vec{r}$ Note: $r_1 + r_2 = r$ since they are antiparallel.

Center of mass frame very convenient for total energy (*KE*+*PE*):

$$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}$$

(Problem 3)

where $r = |\vec{r}|, \vec{v} = \frac{d\vec{r}}{dt}, v = |\vec{v}|$

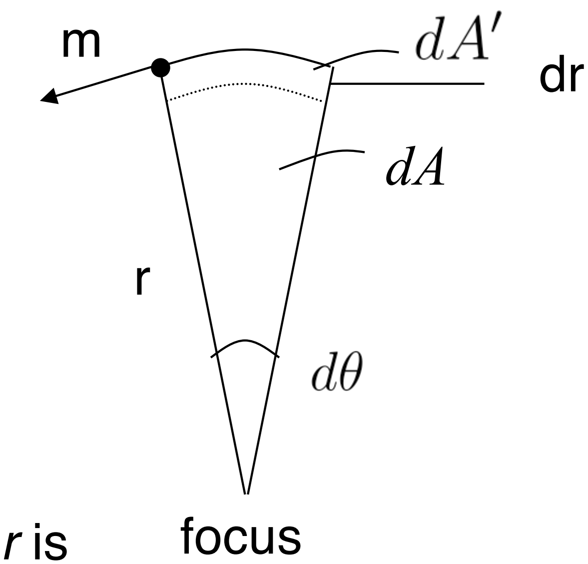
Also, angular momentum $\vec{L} = \mu\vec{r} \times \vec{v} = \vec{r} \times \vec{p}$

=> Orbit is equivalent to reduced mass μ orbiting non-accelerated mass *M* at a distance *r* (with semi-major axis *a*) and speed *v*. Can also show $a = a_1 + a_2$ (Problem 4). C+O show the orbit of μ is a conic section.

Kepler's Second Law Revisited

"A line connecting a planet to the Sun sweeps out equal areas in equal time intervals"

Consider area of ellipse in polar coordinates



$$dA' = dr(r d\theta)$$

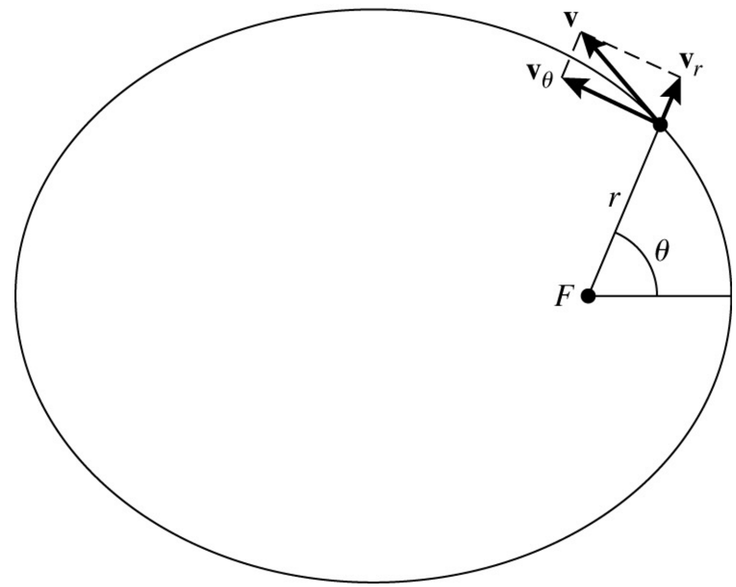
So, integrating over *r*, area from focus to distance *r* is

$$dA = \frac{1}{2} r^2 d\theta$$

as *m* orbits *CM,* it sweeps out this area

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

The orbital velocity *v* can be decomposed:



$$\vec{v} = \vec{v_r} + \vec{v_\theta}$$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta}$$

$$\frac{dA}{dt} = \frac{1}{2}rv_\theta = \frac{1}{2}|\vec{r} \times \vec{v}| = \frac{1}{2}|\vec{r} \times \frac{\vec{p}}{\mu}|$$

$$= \frac{1}{2}\frac{L}{\mu} = const$$

=> Area swept out at constant rate!

Kepler's third law:

It can be shown that the total orbital angular momentum for closed orbits is (understand this in C+O!):

$$L = \mu\sqrt{GMa(1-e^2)}$$

Q: For what shape orbit is L at its maximum? Minimum?

Now integrating:

$$\frac{dA}{dt} = \frac{1}{2}\frac{L}{\mu}$$

is therefore constant, so

$$A = \frac{1}{2}\frac{L}{\mu}P$$

Also, for an ellipse (see prob 2.2.) $A = \pi ab$

Substitute, square and re-arrange:

$$P^2 = \frac{4\pi^2 a^2 b^2 \mu^2}{L^2}$$

Using expressions for *L* above and recalling from lecture 1 that

$$b^2 = a^2(1 - e^2)$$

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3$$

Newton's form of Kepler's third law

Q's: Why did Kepler miss the (m_1+m_2)?

What systems would you observe where you couldn't miss it?

How would you go about weighing stars?

The Virial Theorem

- For an isolated, gravitationally bound system in equilibrium, the total energy (constant) is always one-half the time-averaged potential energy.

$$E = \frac{< U >}{2}$$

- Since $E = <U> + <KE>, \;\; 2<KE> + <U> = 0$

- This is extremely useful, applies to a wide variety of astrophysical situations, whether you have two objects or a large number.

Example: How to weigh a cluster of galaxies?

Assume a spherical cluster with *N* galaxies, with time-average separation *R*, and random time-average speed <*V*>, individual galaxy mass *m*. Note that if in equilibrium, we can assume time average = ensemble average, which we can measure.

$$<K> = \frac{Nm<V>^2}{2}$$

$$< U_{pair} >= -\frac{Gm^2}{R}$$

Number of pairs: N(N-1)/2

$$\frac{Nm<V>^2}{2} = \frac{1}{2}\frac{N(N-1)}{2}\frac{Gm^2}{R}$$

Since N>>1

$$\frac{Nm < V >^2}{2} = \frac{N^2 G m^2}{4R}$$

Total mass

$$M = Nm = \frac{2R < V >^2}{G}$$

Example: We see a velocity dispersion of 670 km/s in the Virgo Cluster which is 3 Mpc in diameter and roughly spherical. What is the mass of the Cluster?

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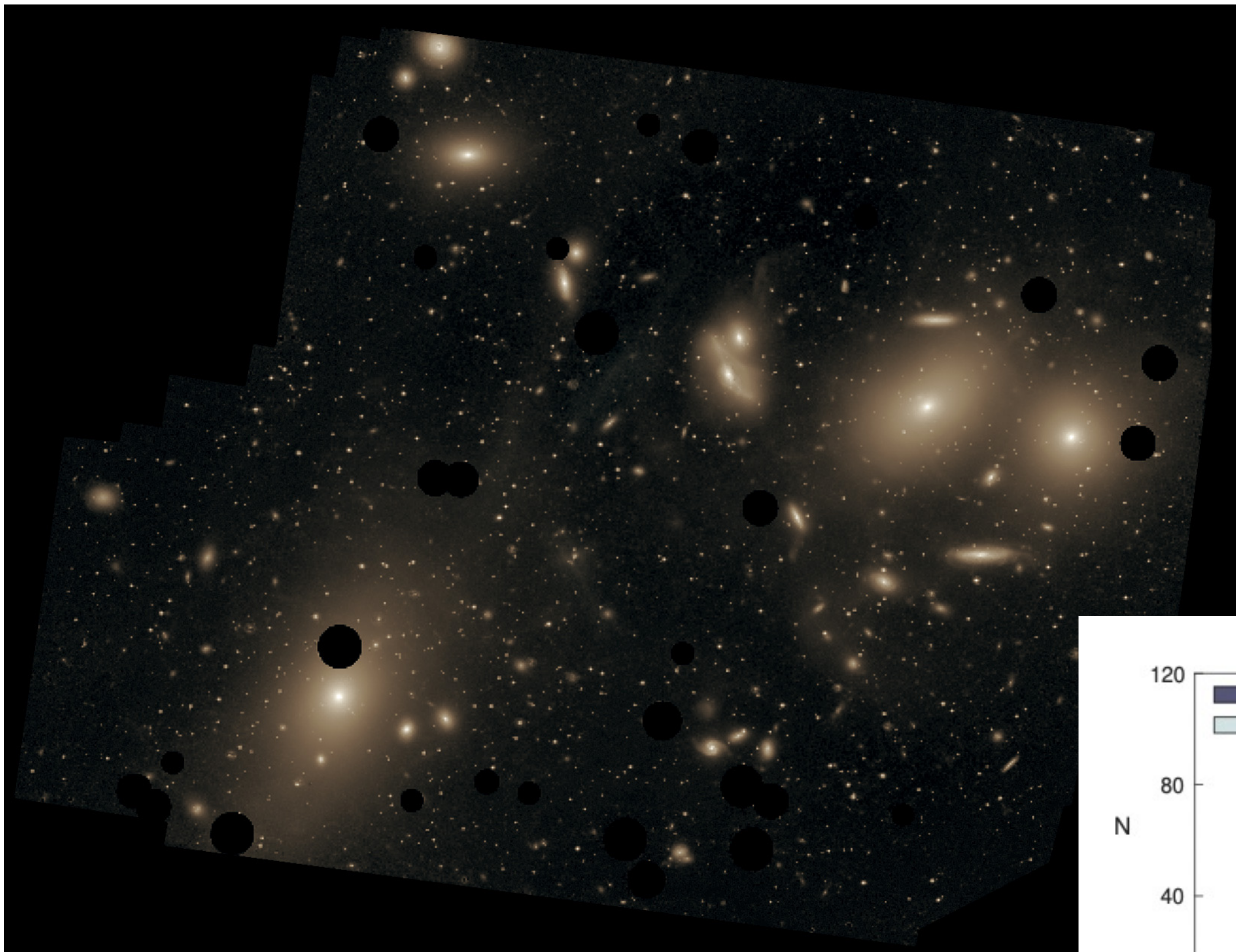
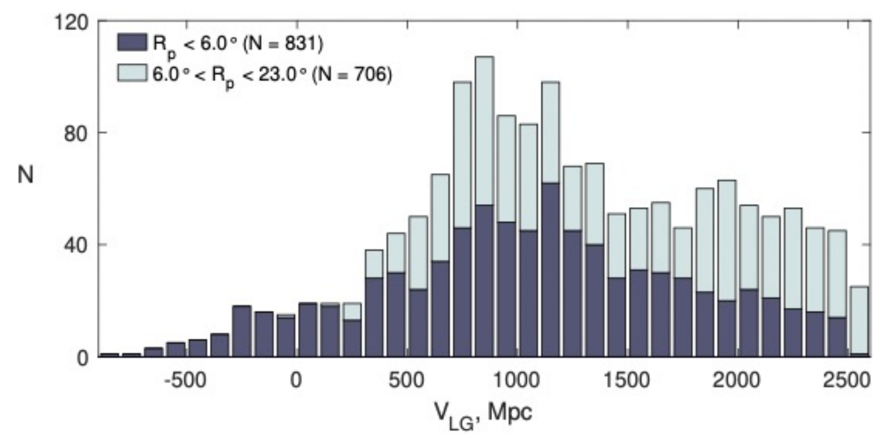


Mihos et al.

Kashibadze et al 2020

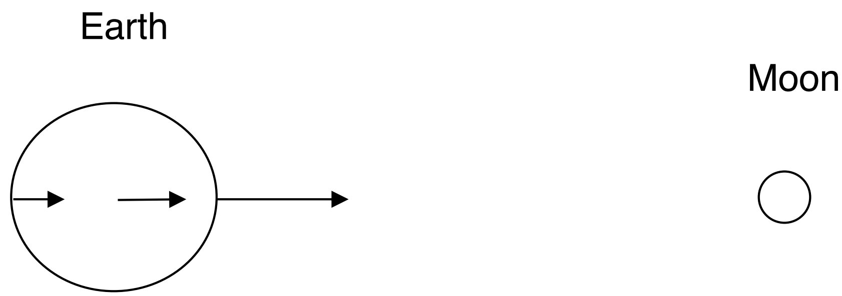


Tidal forces (Sec 19.2)

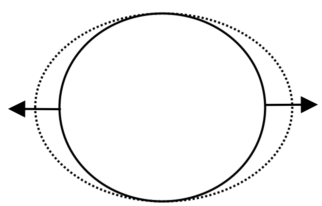
Differential gravitational force = difference between the gravitational forces exerted on two neighboring particles by a third, more distant, body

$$\frac{dF}{dr} = \frac{d}{dr}\left(\frac{GMm}{r^2}\right) \Rightarrow \frac{dF}{dr} = -\frac{2GMm}{r^3}$$

• Tides arise because of the differential gravitational force on opposite sides of the Earth.



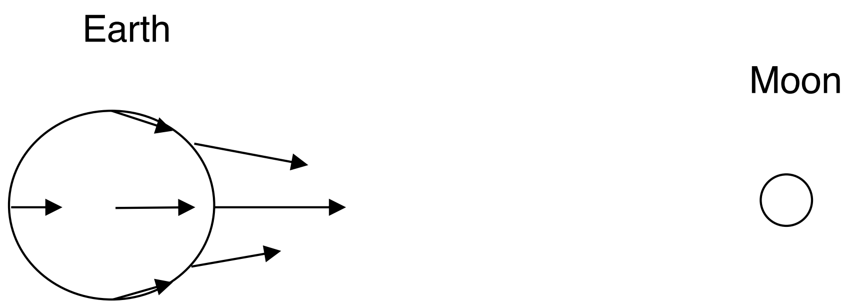
From a "center of Earth" perspective, subtract → from each vector:



Net effect is that for Earth to bulge both towards and away from the Moon.

Q: Why is the Earth's orbital motion determined by the Sun, but the Earth's tides are generated principally by the Moon?

• Not quite the whole story though:



Stretching ll to Earth-Moon axis

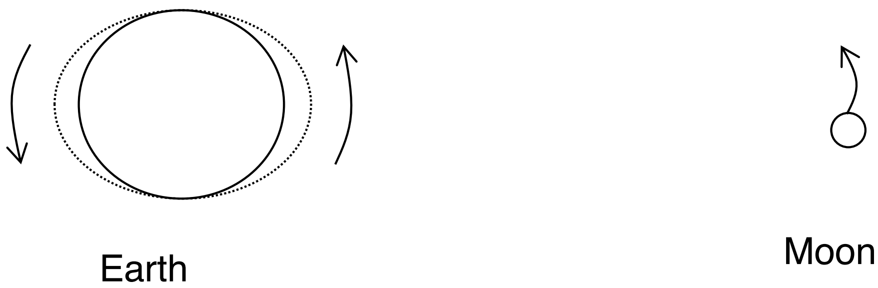
Squeezing $\perp$ to Earth-Moon axis

See C&O 19.2 for a full-blown treatment.

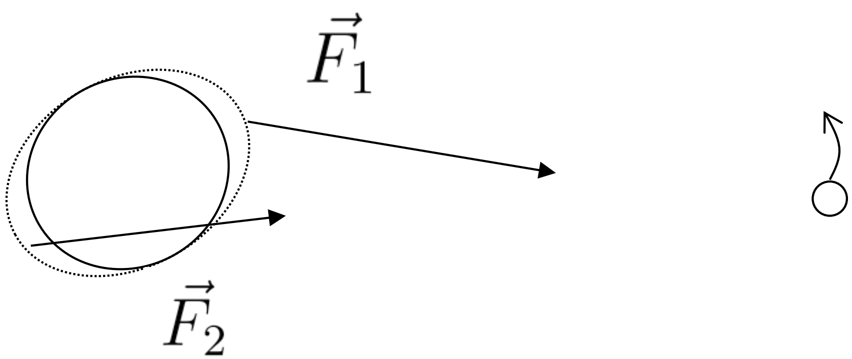
Quantitatively, stretching (dF/dr) is a larger term.

Qualitatively, remember both - black holes.

- **Spin and orbit evolution:**



- Friction => tidal bulge *leads* Earth-Moon line



$$F_1 > F_2$$

$\vec{F}_1 - \vec{F}_2$ has a component opposing Earth's rotation (torque).

- Earth's spin angular momentum changes:

$$\dot{P} = 0.0016\mathrm{s}/\mathrm{century}$$

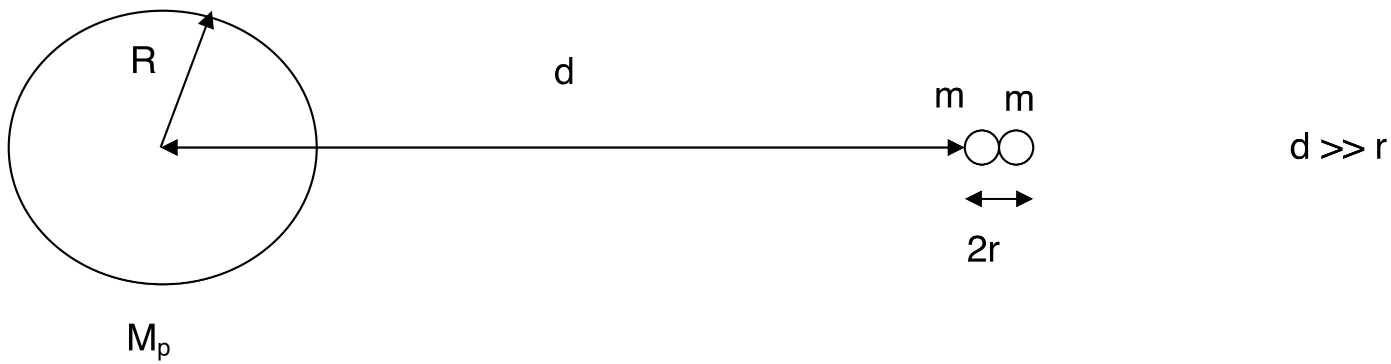
Also, Moon dragged along faster, migrates outward 4 cm/year, gains angular momentum.

Total $\vec{L}$ of system is conserved.

Q: What would happen if the satellite were “leading” the planet's tidal bulge?

• Final point about tidal forces - **The Roche Limit:**

• The distance from a planet below which the differential pull of the planet on two neighboring particles exceeds their mutual gravitation.



• Consider two small masses, each of mass *m*, radius *r*, at a distance *d* from center of the planet with mass M_p.

• At the Roche Limit:

differential attraction by planet = mutual attraction of two masses.

$$\Delta F = \frac{2GM_p m}{d^3} 2r = \frac{Gmm}{(2r)^2}$$

$$\frac{4M_p r}{d^3} = \frac{m}{4r^2}$$

$$d^3 = \frac{16 M_p r^3}{m} = 16[\frac{\frac{4}{3}\pi R^3 \rho_p}{\frac{4}{3}\pi r^3 \rho_s}] r^3 = 16R^3(\frac{\rho_p}{\rho_s})$$

$\rho_p =$ planet density

$\rho_s =$ satellite density

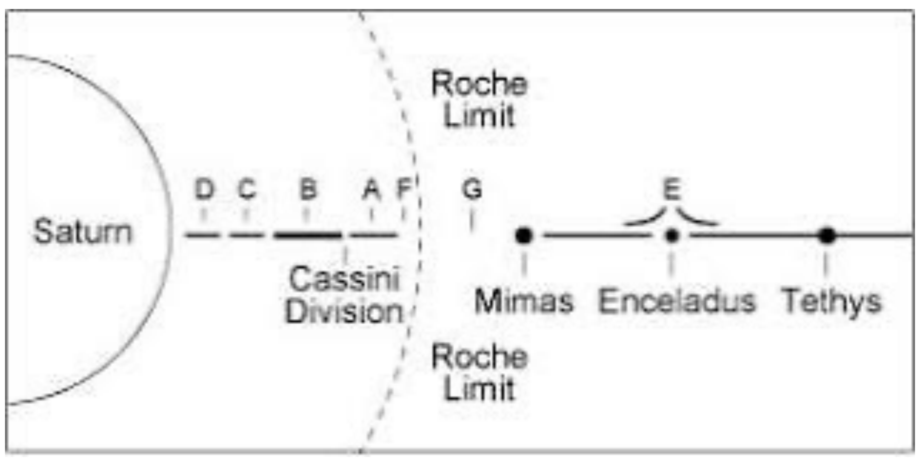
$$\Rightarrow d = \sqrt[3]{16}(\frac{\rho_p}{\rho_s})^{1/3} R \simeq 2.5(\frac{\rho_p}{\rho_s})^{1/3} R$$

- More precise derivation (one spherical body instead of two spheres) gives same but constant is 2.44.
- For gravitationally bound lumps inside Roche Limit, breakup due to tidal forces will occur.

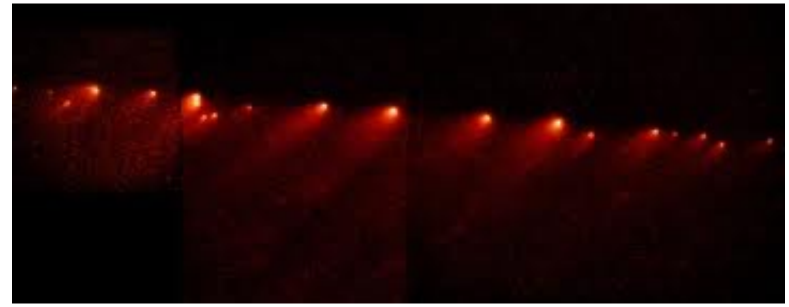
• **Caution:** Satellites, people etc., can live inside Roche limit - why aren't we being pulled to pieces?

Another implication is that the Roche Limit is an accretion limit, outside of which clumps can become gravitationally bound, and inside of which tidal forces will disrupt them.

Are Saturn's rings inside the Roche Limit?



Example: Comet Shoemaker Levy 9.



Passed inside Jupiter's Roche Limit on previous passage. How close would it have to get?

Comet nuclear density ~ 0.5 g cm^{-3}.
Jupiter's density 1.33 g cm^{-3}. Radius 74,500 km.

$$d \simeq 2.5(\frac{\rho_p}{\rho_s})^{1/3} R$$

$d \sim$ 250,000 km

Collision physics

• A very important and powerful means of making quantitative estimates about things with simple arithmetic.

• You can answer questions like:

  – How long before a major city is destroyed by a collision with an asteroid?
  – Do stars collide when galaxies collide?
  – How far does an atom travel between collisions deep in the solar interior?

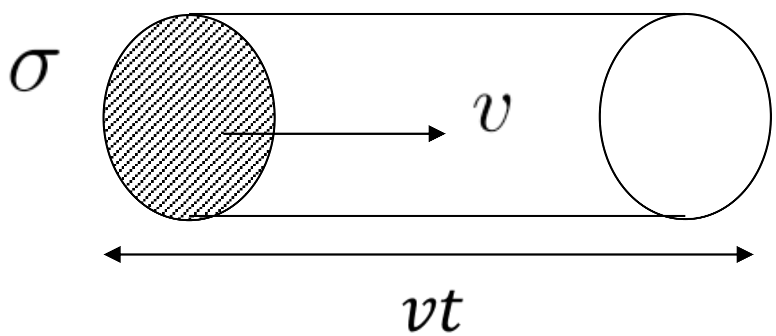
Assume:

A cloud of identical particles, each with radius r and cross section $\sigma=\pi r^2$

Mean number density of particles is n (#/volume)

Typical velocity with respect to cloud of particles is v.

Then in a time t, a particle moving with velocity v sweeps out a volume σvt .



Number of particles in that volume it will collide with is $N = nV = n\sigma vt$

Distance traveled between collisions is $\lambda = \frac{vt}{N} = \frac{vt}{n\sigma vt} = \frac{1}{n\sigma}$

Time between collisions is $\tau = \frac{\lambda}{v} = \frac{1}{n\sigma v}$

λ is mean free path $= \frac{1}{n\sigma}$ τ is mean free time $= \frac{1}{n\sigma v}$

τ is the mean free time a given particle spends between collisions - **not** the average time between *any* collisions in the entire cloud of particles.

Will apply this to how photons move through stellar interiors and atmospheres later.

Worksheet #2

Problem: How long until the Sun collides with another star? Assume the Sun is moving at 220 km/s and the density of stars in the stellar neighborhood stays constant at 0.125 stars/cubic parsec.

How does this compare to the age of the Universe? How likely is it that the Sun will collide with another star?

•An extreme case of tides: picture an iron cube, 1cm on a side held just above a neutron star R=10 km, M=1.4 $M_{\odot}$, density iron 7.86 g cm^{-3}. Assume you divide the cube into two halves, with half of the mass in each part. The tidal stress on the cube can then be estimated as:

$$\frac{F_{lower} - F_{upper}}{(1\mathrm{cm}^2)} \simeq \frac{F_{lower}}{(1\mathrm{cm}^2)}\left(1 - \frac{F_{upper}}{F_{lower}}\right) =$$

$$= \left[\frac{F_{upper}}{F_{lower}} = \frac{F(R+dr)}{F(R)} = \frac{F(R) + \frac{dF}{dR}dr}{F(R)} = 1 + \frac{(-\frac{2GMm}{R^3})dr}{\frac{GMm}{R^2}} = 1 - \frac{2dr}{R}\right] =$$

$$= G\frac{Mm}{R^2}\frac{1}{(1\mathrm{cm}^2)}[1 - (1 - \frac{2dr}{R})] = \frac{2GMmdr}{R^3} = 1.46 \times 10^4 \ \mathrm{Ncm}^{-2}$$

•An iron meteorite falling toward the surface of a neutron star would be stretched (to the point of rupturing) into a thin ribbon.