Sky Noise-Induced Spatial Correlation

Steve Ellingson[*]

October 14, 2008

Contents

[*]Bradley Dept. of Electrical & Computer Engineering, 302 Whittemore Hall, Virginia Polytechnic Institute & State University, Blacksburg VA 24061 USA. E-mail: ellingson@vt.edu

1   Summary

This memo attempts to quantify the correlation between the open-circuit terminal voltages of identical antennas in response to sky noise. Obviously, two identical antennas with zero separation between them produce voltages which are perfectly correlated. Less obvious but equally true (and demonstrated here) is that two antennas separated by many wavelengths produce voltages which can still be suprisingly well-correlated, even if the entire sky is assumed to have uniform brightness.

2   Theory

Consider a patch of sky which is sufficiently small to be modeled as having a uniform brightness temperature $T(\psi)$, where ψ specifies location on the sky (equivalently, direction of incidence). By the Rayleigh-Jeans Law, the associated flux is

$$\Delta S(\psi) = \frac{2k}{\lambda^2} T(\psi) \Delta \Omega \tag{1}$$

where k is Boltzmann's constant, λ is wavelength, $\Delta \Omega$ is the solid angle covered by the patch. The units of flux are W m^{-2} Hz^{-1}. The incident electric field which gives rise to this flux is a plane wave $\Delta \mathbf{E^i}(\psi)$, thus

$$\Delta S(\psi) = \frac{\left| \Delta \mathbf{E^i}(\psi) \right|^2}{2\eta} \tag{2}$$

where η is the impedance of free space (≈ 377 ohms), the units of the electric field are V m^{-1} Hz^{-1} where the root-mean-square (RMS) voltage is assumed. The flux is, to a reasonable approximation, equally divided between the two orthogonal polarizations of the incident electric field. Thus:

$$\left| \Delta \mathbf{E^i}_\theta(\psi) \right| = \left| \Delta \mathbf{E^i}_\phi(\psi) \right| = \sqrt{\eta \Delta S(\psi)} \tag{3}$$

where the subscripts θ and ϕ are used to denote the corresponding linear polarizations of the incident field. Here, we use a coordinate system which has its origin at the terminals of an arbitrarily-selected reference antenna and in which Cartesian axes x, y, and z correspond to East, North, and the zenith, respectively. We shall also use an associated spherical coordinate system in which θ is the angle measured from the zenith toward the horizon, and ϕ is the angle measured from East toward North. The position of the nth antenna is

$$\mathbf{p}_n = \hat{\mathbf{x}} x_n + \hat{\mathbf{y}} y_n \tag{4}$$

and the unit vector $\hat{\mathbf{r}}$, pointing from the origin toward the patch of sky indicated by ψ, is

$$\hat{\mathbf{r}}(\psi) = \hat{\mathbf{x}} \cos\phi \sin\theta + \hat{\mathbf{y}} \sin\phi \sin\theta + \hat{\mathbf{z}} \cos\theta \tag{5}$$

The two polarizations of the electric field incident on the nth antenna can now be expressed as

$$\Delta \mathbf{E^i}_{\theta,\mathbf{n}}(\psi) = \hat{\theta} \; \mathbf{e}(\psi) \frac{\sqrt{2\mathbf{k}\eta}}{\lambda} \sqrt{\mathbf{T}(\psi)\boldsymbol{\Delta\Omega}} \; \mathbf{e}^{-\mathbf{j}\beta \mathbf{d_n}(\psi)} \tag{6}$$

$$\Delta \mathbf{E^i}_{\phi,\mathbf{n}}(\psi) = \hat{\phi} \; \mathbf{e}(\psi) \frac{\sqrt{2\mathbf{k}\eta}}{\lambda} \sqrt{\mathbf{T}(\psi)\boldsymbol{\Delta\Omega}} \; \mathbf{e}^{-\mathbf{j}\beta \mathbf{d_n}(\psi)} \tag{7}$$

where $\beta = 2\pi/\lambda$,

$$d_n(\psi) = -\mathbf{p}_n \cdot \hat{\mathbf{r}}(\psi) \; , \tag{8}$$

and $e(\psi)$ is a zero-mean complex-valued Gaussian-distributed random variable with unit variance.

The open-circuit voltage generated at the terminals of this antenna when illuminated by this electric field is

$$\Delta v_n(\psi) = \left[\Delta \mathbf{E^i}_{\theta,\mathbf{n}}(\psi) + \Delta \mathbf{E^i}_{\phi,\mathbf{n}}(\psi) \right] \cdot \mathbf{l}_e(\psi) \tag{9}$$

where $\mathbf{l}_e(\psi)$ is the *vector effective length* of the antenna, having units of m. We shall assume that all antennas have the same vector effective length; i.e., are identical and identically-oriented. The total voltage, now accounting for the entire sky, is:

$$V_n = \sum_\psi \Delta v_n(\psi) \tag{10}$$

The correlation between two antennas "m" and "n" is

$$\rho_{mn} = \langle V_m V_n^* \rangle = \int V_m V_n^* dt \tag{11}$$

where the integration is really operating only on the random variable $e(\psi)$. Note that the product $V_m V_n^*$ consists of the sum of *self-terms* having the form $v_m(\psi)v_n^*(\xi)$ where $\psi = \xi$, and *cross-terms* having the same form but with $\psi \neq \xi$. The cross-terms go to zero in the correlation because $e(\psi)$ is uncorrelated with $e(\xi)$, so only the self-terms remain. Thus we have

$$\rho_{mn} = \sum_\psi \langle \Delta v_m(\psi) \; \Delta v_n^*(\psi) \rangle \tag{12}$$

which, after substitutions and some algebra, becomes

$$\rho_{mn} = \frac{2k\eta}{\lambda^2} \sum_\psi \left| \left(\hat{\theta} + \hat{\phi} \right) \cdot \mathbf{l}_e(\psi) \right|^2 \; e^{-j\beta(d_m(\psi) - d_n(\psi))} \; T(\psi)\Delta\Omega \tag{13}$$

Note that this correlation can itself be expressed in terms of an antenna temperature. Using a Thevenin equivalent circuit model, a receiving antenna can be modeled as the open circuit voltage V_n in series with the antenna's self-impedance, Z_A. The power delivered by this antenna into a load Z_L is $P_L = \mathrm{Re}\{V_L I_L^*\}$, where V_L and I_L are the RMS voltage and current through the load, which appears in series with the antenna impedance. Thus:

$$P_L = \mathrm{Re}\left\{ \left(V_n \frac{Z_L}{Z_A + Z_L} \right) \; \left(\frac{V_n}{Z_A + Z_L} \right)^* \right\} = \rho_{nn} \frac{R_L}{|Z_A + Z_L|^2} \tag{14}$$

The antenna temperature T_A can be defined as the power delivered by the antenna to a conjugate-matched load divided by k. Thus, using $Z_L = Z_A^*$, we find:

$$T_A = \frac{\rho_{nn}}{4kR_A} \tag{15}$$

Finally, we note that we can generalize this definition to obtain a "correlation temperature" between any two antennas having the same impedance:

$$T_A^{(mn)} = \frac{\rho_{mn}}{4kR_A} \tag{16}$$

which equals T_A when $m = n$; and can be complex-valued for $m \neq n$.

3   Examples

We now consider several examples. First, let us consider the case of straight half-wave dipoles in free space; i.e., no ground. Let us assume the antenna is oriented North-South; i.e., along the y-axis in the local coordinate system. In free space, it is known that the vector effective length would be:

$$\mathbf{l}_e(\psi) = \hat{\mathbf{y}} l_{e0} \tag{17}$$

where $l_{e0} = \pi\lambda/8$. The frequency is 38 MHz ($\lambda = 8$ m), and we assume an artificially-homogeneous sky at a uniform brightness temperature of 9000 K. The self-impedance of the dipole R_A is determined from the resulting self-correlation temperature, using Equation 15, and found to be 121.8 ohm. Mutual coupling is ignored. The resulting correlation temperature as a function of separation between two identical antennas is shown in Figure 1. Note that the correlation falls off much more rapidly with separation along the y-oriented baseline than for the x-oriented baseline, presumably because the antenna pattern is much narrower in the E-plane than in the H-plane. Also notable is that nulls occur every half-wavelength, which is consistent with a well-known result from propagation theory. In this scale, the diameter of an LWA station (assuming it is 100 m) is 12.5λ, and total horizontal span is 160 m.

Next, we add an infinite and perfectly conducting ground screen, and assume the dipoles are located $\lambda/4$ above it. A modification that accounts for the presence of the ground is:

$$\mathbf{l}_e(\psi) \approx \hat{\mathbf{y}} 2 \left(1 - \cos\phi\sin^2\theta \right)^{0.94} l_{e0} \tag{18}$$

where the factor of 2 accounts for the doubling of gain in the direction of the zenith due to ground reflection, and the remaining factor introduces a $\cos^{1.88}\theta$ dependence in the H-plane. Note that this makes the antenna pattern much narrower, especially in the H-plane. The antenna self-impedance R_A is found in this case to be 1335.4 ohm. Using again the 38 MHz, 9000K artificially-homogeneous sky, the result is shown in Figure 2. Again the correlation falls off much more rapidly with separation along the y-oriented baseline than for the x-oriented baseline, with the former now rolling off considerably faster. The periodic nulling is now mostly suppressed, which is probably due to the narrowing of the antenna pattern.

Finally, we introduce a realistic sky, illustrated in Figure 3. The brightness temperatures were obtained using the low-frequency sky model described in [1]. In this case, the southwestern horizon region is much brighter than other parts of the sky, although the overall antenna temperature is not much different from 9000 K, as can be verified from the zero-separation value shown in Figure 4. Compared to the artificial (uniformly bright) sky, we now see the correlation is much larger for large separations along a north-south baseline, and the half-wavelength null spacing has returned. Correlation along the east-west baseline is not much affected. Figure 5 shows the same experiment computed for much longer baselines; up to 800 m.

References

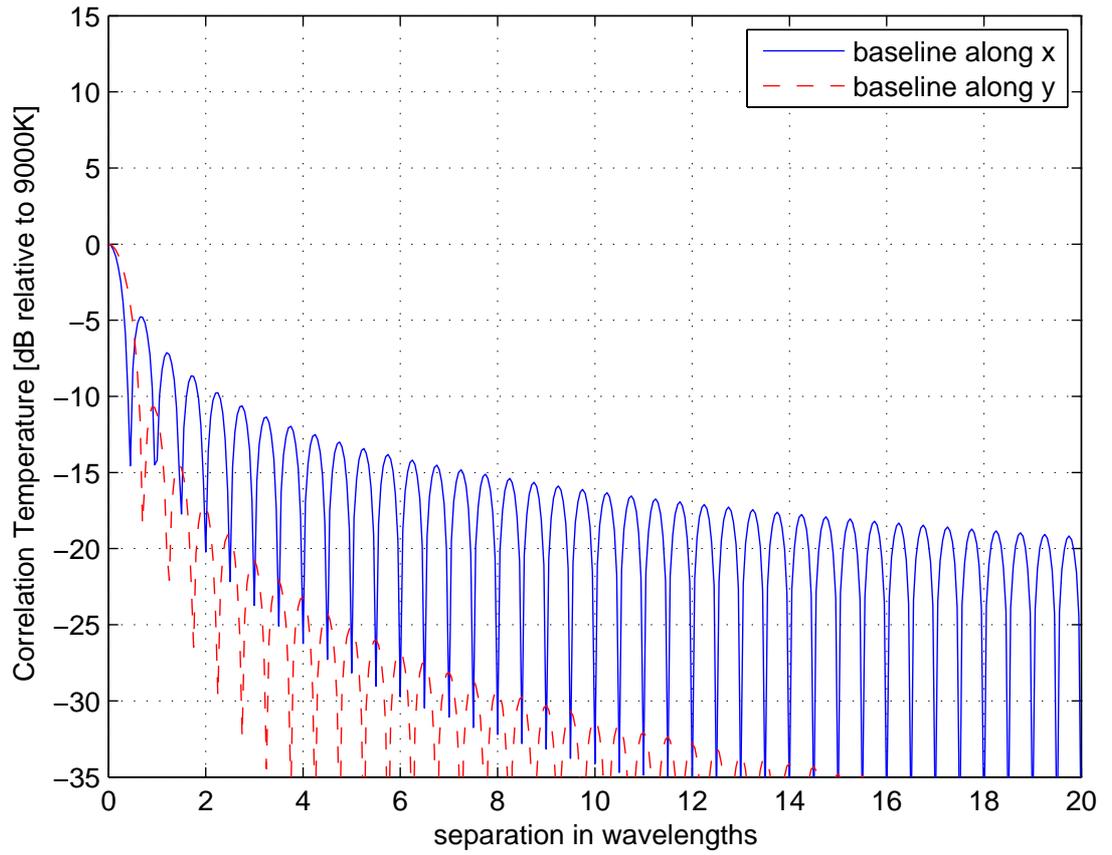[1] E. Polisensky, "LFmap: A Low Frequency Sky Map Generating Program", Long Wavelength Array Memo Series No. 111, Sep 7, 2007. [online] http://www.phys.unm.edu/~lwa/memos.

Figure 1: Correlation temperature between two identical and identically-oriented ($\hat{\mathbf{y}}$) half-wave dipoles as a function of separation between them: No ground, uniform 9000 K sky.
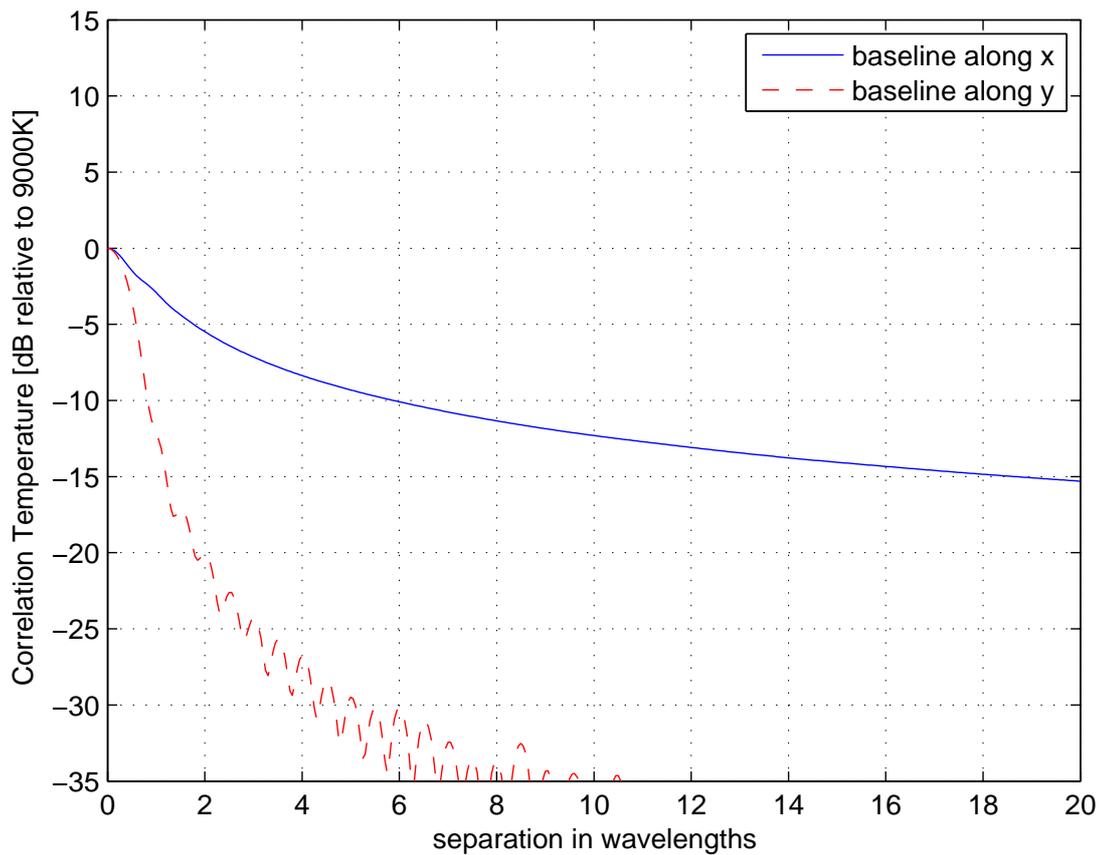
Figure 2: Correlation temperature between two identical and identically-oriented ($\hat{\mathbf{y}}$) half-wave dipoles as a function of separation between them: Infinite perfectly conducting ground, uniform 9000 K sky.
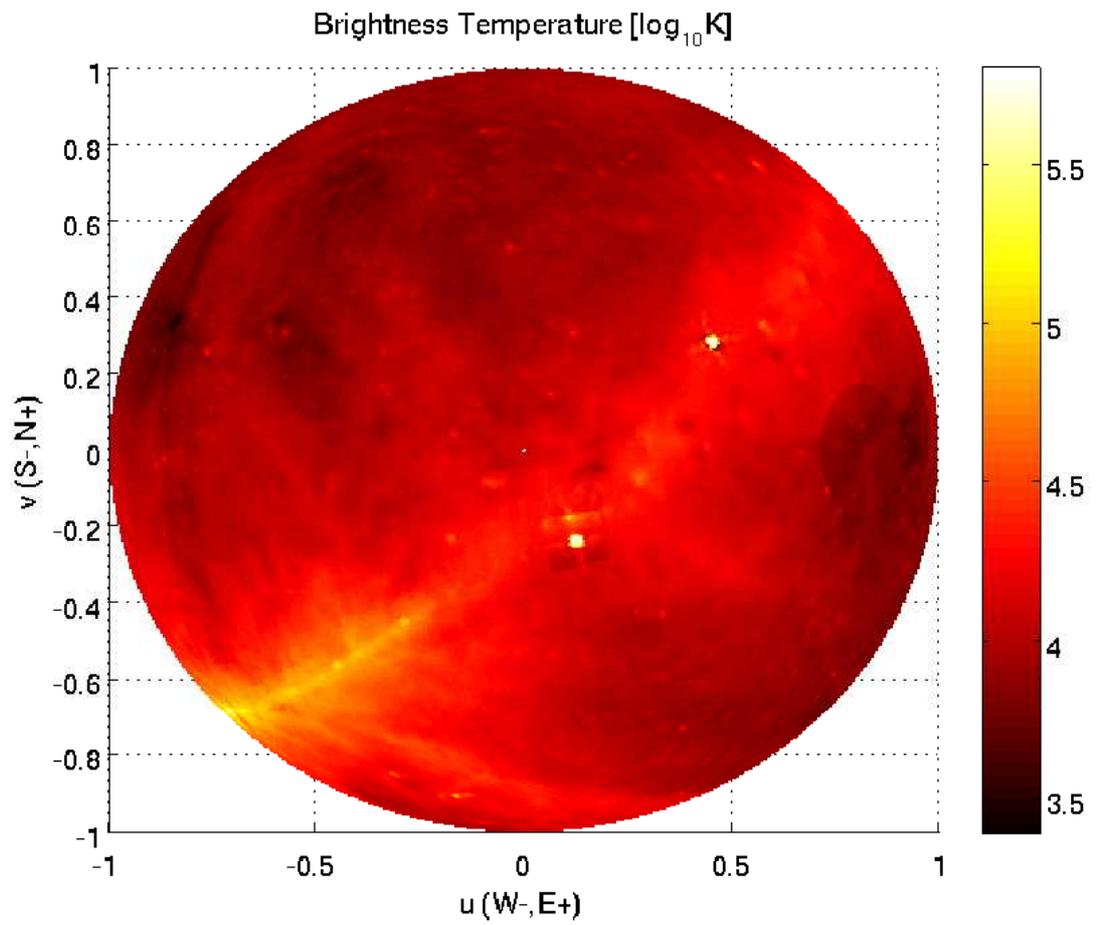
Figure 3: Realistic sky model. 38 MHz. Latitude 35° N, time chosen to correspond to a scene in which the sky brightness is far from uniform.
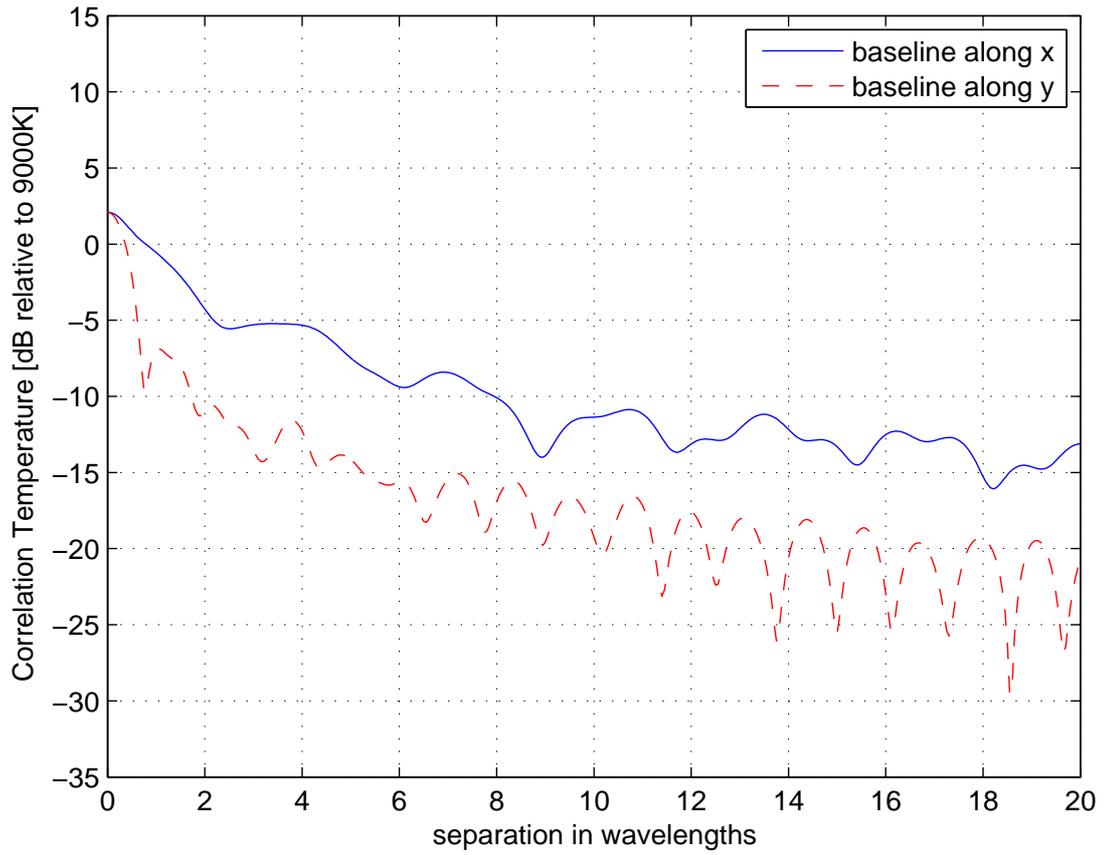
Figure 4: Correlation temperature between two identical and identically-oriented ($\hat{\mathbf{y}}$) half-wave dipoles as a function of separation between them: Infinite perfectly conducting ground, realistic sky.
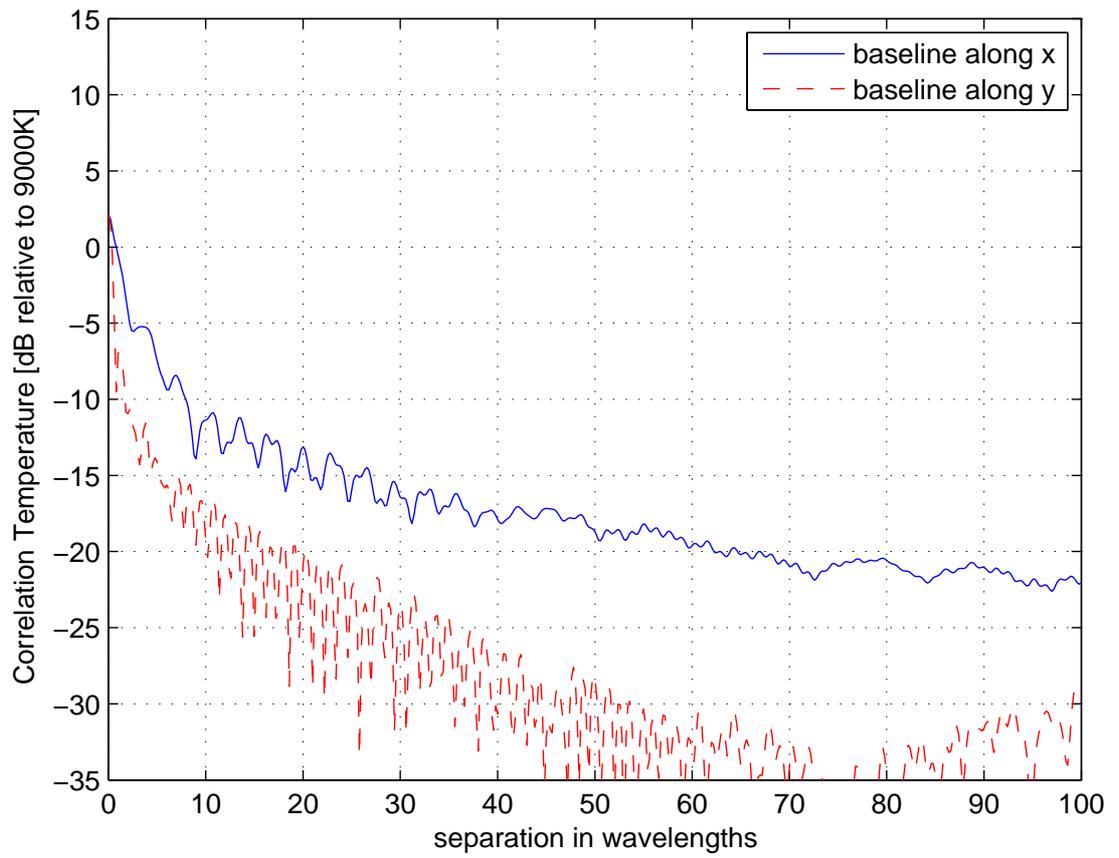
Figure 5: Same as Figure 4, except now extending to much longer baselines.